JOURNAL OF ALGEBRA 115, 303–312 (1988)

# Duality Theorems for Rings with Actions or Coactions

MARGARET BEATTIE

*Department of Mathematics and Computer Science, Mount Allison University, Sackville, New Brunswick, Canada E0A 3C0*

*Communicated by I. N. Herstein*

Received July 15, 1986

Suppose $A$ is an algebra with unit over a commutative ring $k$. Then it is well known that a group $G$ acts as a group of ring automorphisms on $A$ if and only if $A$ is a $k[G]$-module algebra where the group ring $k[G]$ has the usual Hopf algebra structure. Also, $A$ is a $G$-graded ring if and only if $A$ is a $k[G]$-comodule algebra. If $G$ is finite, then $k[G]^*$, the dual of $k[G]$, is also a Hopf algebra, and $A$ is a $k[G]$-comodule algebra if and only if $A$ is a $k[G]^*$-module algebra. Cohen and Montgomery have shown in [8] that for $G$ finite of order $n$, if $A$ is a $k[G]$-module algebra, then

$$(A \# k[G]) \# k[G]^* \cong M_n(A),$$

and if $A$ is a $k[G]^*$-module algebra,

$$(A \# k[G]^*) \# k[G] \cong M_n(A).$$

If $G$ is infinite, however, $k[G]^*$ is no longer a Hopf algebra.

In this note, we extend the definition of $k[G]^*$ to infinite $G$ and prove duality theorems analogous to those of Cohen and Montgomery by using the Morita theory for rings with local units developed by Anh and Marki [2]. We let $G * A$ denote our generalization of $A \# k[G]^*$; the rings coincide if $G$ is finite. For infinite $G$, $G * A$ is a nonunitary subring of the ring $\tilde{A} \# G$ defined by Quinn [10] in his discussion of duality and is an essential left and right ideal of $\tilde{A} \# G$. $G$ acts as a group of ring automorphisms on $G * A$ with $(G * A)^G \cong A$ if $G$ is finite and (0) otherwise. Futhermore, $G * A$ has the nice property that the category of left (right) unitary $G * A$-modules is isomorphic to the category of left (right) $G$-graded $A$-modules.

To prove our first duality theorem, we define a Morita context for a graded ring $A$; this Morita context is strict if and only if $A$ is strongly $G$-



0021-8693/88 $3.00

graded. For $S$ a ring on which $G$ acts as a group of automorphisms, let $A$ be $S * G$, the skew group ring; then, from our Morita context, we obtain

$$G * (S * G) \cong M_G(S)^{\text{fin}},$$

where $M_G(S)^{\text{fin}}$ is the ring of matrices over $S$ with rows and columns indexed by $G$ and with only finitely many nonzero entries. Note that if $S$ is a field, then $M_G(S)^{\text{fin}}$ is dense in $\text{End}_S(S * G)$ with the finite-discrete topology [5].

To obtain our second duality theorem, namely that for $R$ a $G$-graded ring,

$$(G * R) * G \cong M_G(R)^{\text{fin}},$$

we first define a strict Morita context with the rings $(G * R) * G$ and $R$ and then use the arguments of the first section to obtain the duality statement. If $G$ is finite so that $G * R$ has a unit and $(G * R)^G \cong R$, then this Morita context is that of Cohen in [6]. This second duality result was also proved by Quinn [10] by a different method.

## Preliminaries

Throughout, $G$ will denote a group with identity $e$. A ring $A$ is called a $G$-graded ring if $A = \bigoplus_{g \in G} A_g$ and $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$. If $A_g A_h = A_{gh}$ for all $g, h \in G$, then $A$ is called strongly $G$-graded. By [9, Lemma 1.3.2], if $A$ has a unit, $A$ is strongly $G$-graded if and only if $R_g R_{g^{-1}} = R_e$ for all $g \in G$. An element $a$ in $A_g$ is called homogeneous of grade $g$ and may be denoted $a = a_g$. Any element $a$ in $A$ may be written uniquely as $a = \sum_{g \in G} a_g$ with $a_g \in A_g$ and only finitely many homogeneous components $a_g$ nonzero. A left $A$-module $M$ is called a graded left $A$-module if $M = \bigoplus_{g \in G} M_g$ and $A_g M_h \subseteq M_{gh}$. The category $A$-gr is the category of graded left $A$-modules with morphisms the graded morphisms of degree $e$, i.e., the grade-preserving $A$-module homomorphisms. Further details on graded rings may be found in [9].

## 1. A Morita Context for Graded Rings and a Duality Theorem for Group Actions

Let $R$ be a $G$-graded ring with unit. We begin by defining $G * R$, our candidate for a ring to play the role that the skew group ring plays for rings with $G$-action.

DEFINITION 1.1. For $R$ a $G$-graded ring, define $G * R$ to be the ring whose elements are (finite) sums of elements of the form $xv_g$, $x \in R$, $g \in G$ with

$$xv_g + yv_g = (x+y)\,v_g \qquad \text{and} \qquad (xv_g)(yv_h) = xy_{gh^{-1}}v_h.$$

The definition of multiplication then extends as usual to finite sums. We write $v_g$ for $1v_g$. Note that $v_g v_h = v_g$ if $g = h$ and 0 otherwise.

It is easy to check that this definition gives an associative ring. If $G$ is finite, then $\sum_{g \in G} v_g$ is a unit in $G * R$; otherwise $G * R$ is a ring without a unit. For finite $G$, $G * R$ is the same as $R \# k[G]^*$ and $\tilde{R} \# G$, where $R \# k[G]^*$ is the smash product discussed in [8] (here we need $R$ an algebra over a commutative ring $k$) and $\tilde{R} \# G$ is the ring defined for finite or infinite $G$ in [10]. For infinite $G$, $G * R$ is the nonunitary subring $\bigoplus \tilde{R}p(x)$ of $\tilde{R} \# G$. $G *$ - is a covariant functor from the category of $G$-graded rings and grade-preserving ring homomorphisms to rings.

Note that $G * R$ is a ring with local units in the sense of either [1] or [2]. For every finite subset $T$ of $G * R$ is contained in a subring of the form $f(G * R)f$, where $f$ is an idempotent in $G * R$; let $f$ be the sum of the $v_g$ such that either $v_g x$ or $xv_g$ is nonzero for some $x \in T$. Since $T$ is finite, this sum is finite, $f$ is the required idempotent, and such idempotents commute.

A left $G * R$-module $M$ is called unitary if $(G * R)M = M$. It is straightforward to show that $M$ is unitary if and only if for each finite subset $T$ of $M$, there exists an idempotent $f$ in $G * R$ such that $fm = m$ for all $m \in T$. Thus every submodule of a unitary left $G * R$-module is unitary. Similar statements hold for right modules. Details may be found in [2] or [4].

It is shown in [4] that the group $G$ acts as a group of ring automorphisms on $G * R$ on the left and right by

$$g(xv_h) = {}^g(xv_h) = xv_{hg^{-1}} \qquad \text{and} \qquad (xv_h)^g = xv_{hg}.$$

If $G$ is finite, then $(G * R)^G = R$; otherwise $(G * R)^G = (0)$. It is also shown in [4] that the categories of unitary left (right) $G * R$-modules and graded left (right) $R$-modules are isomorphic.

We now construct a Morita context $\{G * R,\ P = {}_{G*R}R_{R_e},\ Q = {}_{R_e}R_{G*R},\ R_e,\ [\ ,\ ],\ (\ ,\ )\}$. In order to use the Morita theory for rings with local units developed in [2], we must show that $P$ and $Q$ are unitary bimodules and that

$$[\ ,\ ]\colon P \otimes_{R_e} Q \to G * R \qquad \text{and} \qquad (\ ,\ )\colon Q \otimes_{G*R} P \to R_e$$

satisfy the usual linearity and compatibility conditions [3].

Since $R_e$ is a unitary subring of $R$, $R$ is a left and right unitary $R_e$-module. Also, by the category isomorphism mentioned above, since $R$ is a $G$-graded left and right $R$-module, $R$ is a left and right unitary $G * R$-module. In particular, the left and right $G * R$-module structures of $R$ are given by

$$(xv_g)\, y = xy_g \qquad \text{and} \qquad x(yv_g) = (xy)_{g^{-1}}.$$

Then $R$ is a $G * R$-$R_e$- and an $R_e$-$G * R$-bimodule since

$$((xv_g)\, y)\, z = (xy_g)\, z = x(yz)_g = (xv_g)(yz) \qquad \text{for } x, y \in R,\ z \in R_e,$$

and

$$x(y(zv_g)) = x(yz)_{g^{-1}} = (xyz)_{g^{-1}} = (xy)(zv_g) \qquad \text{for } y, z \in R,\ x \in R_e.$$

Now, for $x, y \in R$, define $[\ ,\ ]: R \otimes_{R_e} R \to G * R$ by $[x, y] = \sum_{g \in G} (xv_e)(y_{g^{-1}} v_g) = \sum_{g \in G} xy_{g^{-1}} v_g$, and define $(\ ,\ ): R \otimes_{G * R} R \to R_e$ by $(x, y) = (xy)_e$. Note that $[\ ,\ ]$ and $(\ ,\ )$ are the maps defined in [8] for $G$ finite. We must show that $[\ ,\ ]$ and $(\ ,\ )$ satisfy the necessary linearity and associativity conditions. We show that $[\ ,\ ]$ is left $G * R$-linear. The remaining computations to check linearity are similar; for finite $G$, they may be found in [8]. For $x, y, z \in R$,

$$\begin{aligned}
[(xv_g)\, y, z] &= [xy_g, z] \\
&= \sum_{h \in G} xy_g z_{h^{-1}} v_h \\
&= \sum_{h \in G} x(yz_{h^{-1}})_{gh^{-1}} v_h \\
&= (xv_g) \sum_{h \in G} yz_{h^{-1}} v_h \\
&= (xv_g)[y, z],
\end{aligned}$$

so that $[\ ,\ ]$ is left $G * R$-linear. Also

$$[x, y]\, z = \left( \sum_{g \in G} xy_{g^{-1}} v_g \right) z = \sum_{g \in G} xy_{g^{-1}} z_g = x(yz)_e = x(y, z)$$

and

$$x[y, z] = x \sum_{g \in G} yz_{g^{-1}} v_g = \sum_{g \in G} (xyz_{g^{-1}})_{g^{-1}} = (xy)_e z = (x, y)\, z,$$

so that the compatibility conditions are satisfied.

Therefore we have

THEOREM 1.2. *The sextuple* $\{G * R, {}_{G*R}R_{R_e}, {}_{R_e}R_{G*R}, R_e, [\ ,\ ], (\ ,\ )\}$ *is a Morita context. The Morita context is strict if and only if $R$ is strongly $G$-graded.*

*Proof.* From the discussion above, the given sextuple is a Morita context so we need prove only the last statement. By [2, Theorem 2.2], the functors $R \otimes_{R_e}$-: $R_e$- Mod $\to G * R$-Mod and $R \otimes_{G*R}$-: $G * R$-Mod $\to R_e$-Mod are inverse equivalences of categories if and only if $[\ ,\ ]$ and $(\ ,\ )$ are surjective, where the module categories above are categories of unitary left modules. Since $R$ has a unit element, $(\ ,\ )$ is surjective and we must show that $[\ ,\ ]$ is surjective if and only if $R$ is strongly $G$-graded.

Suppose $[\ ,\ ]$ is surjective and suppose $v_g = \sum_{i=1}^n [x_i, y_i] = \sum_{i=1}^n \sum_{h\in G} x_i y_{i_h^{-1}} v_h$. Then $\sum_{i=1}^n x_i y_{i_g^{-1}} = 1$ so that $\sum_{i=1}^n x_{i_g} y_{i_g^{-1}} = 1$ and $R_g R_{g^{-1}} = R_e$.

Conversely, suppose $R$ is strongly $G$-graded. Then for all $g, h \in G$, $x \in R_g$, since $R_{gh} R_{h^{-1}} = R_g$, there exist $x_i \in R_{gh}$, $y_i \in R_{h^{-1}}$ such that $\sum_{i=1}^n x_i y_i = x$. Then $\sum_{i=1}^n [x_i, y_i] = \sum_{i=1}^n \sum_{t\in G} x_i y_{i_t^{-1}} v_t = \sum_{i=1}^n x_i y_i v_h = x v_h$, and $[\ ,\ ]$ is onto.

Now, let $S$ be a ring with unit on which $G$ acts as a group of ring automorphisms. Then we may form the skew group ring $S * G$ defined in the usual way as sums of elements $s * g$, $s \in S$, $g \in G$ with multiplication of these generators defined by $(s * g)(r * h) = sg(r) * gh$. $S * G$ is a strongly $G$-graded ring with $(S * G)_g = S * g$. Therefore by Theorem 1.2, the Morita context $\{G * (S * G), P = {}_{G*(S*G)}S * G_S, Q = {}_S S * G_{G*(S*G)}, S, [\ ,\ ], (\ ,\ )\}$ is strict. We use this Morita context to show that $G * (S * G)$ is isomorphic to a ring of matrices over $S$, i.e., to prove a duality theorem for group actions.

First, we need some definitions and notation from [2]. If $A$ and $B$ are rings with local units and ${}_B N$, ${}_B M_A$ are unitary (bi-)modules as indicated, then $\mathrm{Hom}_B(M, N)$ is a left $A$-module by putting, for $\phi \in \mathrm{Hom}_B(M, N)$, $a \in A$, $m \in M$, $a\phi$: $m \to (ma)\phi$, where the action of $\phi$ is written on the right. The notation $A\ \mathrm{Hom}_B(M, N)$ means the largest unitary $A$-submodule of $\mathrm{Hom}_B(M, N)$.

THEOREM 1.3. $G * (S * G) \cong M_G(S)^{\mathrm{fin}}$, *where $M_G(S)^{\mathrm{fin}}$ denotes the ring of matrices over $S$ with rows and columns indexed by $G$ and with finitely many nonzero entries.*

*Proof.* Since the Morita context above is strict, by [2, Theorems 2.1 and 2.2], $G * (S * G) \cong U = G * (S * G)\ \mathrm{End}_S(S * G)$, where $U$ is the largest unitary left $G * (S * G)$-submodule of $\mathrm{End}_S(S * G)$.

Since $S * G = \bigoplus_{g\in G} (S * g)$, elements of $\mathrm{End}_S(S * G)$ may be written as matrices over $S$ with rows and columns indexed by $G$ and with the action of $\phi \in \operatorname{End}_S(S * G)$ being matrix multiplication on the right. The $g$th row of the matrix associated with $\phi$ is determined by the image of $1 * g$ under $\phi$, and so has only finitely many nonzero entries. Thus $\operatorname{End}_S(S * G)$ is the ring of row finite matrices over $S$.

Suppose $\phi \in U$. Then since $U$ is unitary, $\phi = (\sum_{h \in F} v_h)\, \phi$, where $F$ is a finite subset of $G$. Then for any $g \in G$, $(1 * g)\, \phi = ((1 * g) \sum_{h \in F} v_h)\, \phi = (\sum_{h \in F} (1 * g)_{h^{-1}})\, \phi$, which is zero if $g^{-1} \notin F$. Therefore only finitely many rows in the matrix associated with $\phi$ are nonzero and $\phi \in M_G(S)^{\text{fin}}$.

Also, $M_G(S)^{\text{fin}}$ is a unitary $G * (S * G)$-submodule of $\operatorname{End}_S(S * G)$. If $\phi \in M_G(S)^{\text{fin}}$, let $T = \{g\colon \text{the } g\text{th row of } \phi \text{ is nonzero}\}$. Then $\phi = (\sum_{g \in T} v_g)\, \phi$, and $M_G(S)^{\text{fin}} = U$.

The next section gives a similar proof of a duality theorem for coactions.

## 2. A Duality Theorem for Group Graded Rings

Our aim in this section is to present a theorem describing duality for group coactions (gradings) using Morita theory arguments as in Section 1. Note that the isomorphism $(G * R) * G \cong M_G(R)^{\text{fin}}$ which we prove in Theorem 2.2 is implicit in [10, Lemma 2.2(iv)] since $\tilde{R} \# G = \tilde{R} \oplus [\bigoplus_{x \in G} \tilde{R}p(x)]$, where $\tilde{R}$ is an isomorphic copy of $R$ and $\bigoplus_{x \in G} \tilde{R}p(x) \cong G * R$.

Let $R$ be a $G$-graded ring with unit. Then $G * R$ is an associative ring on which $G$ acts as a group of automorphisms, so that we may form the skew group ring $(G * R) * G$, generated by $xv_g * h$, $x \in R$, $g, h \in G$, with $(xv_g * h)(yv_p * q) = (xv_g)(yv_{ph^{-1}}) * hq$.

Lemma 2.1. *$(G * R) * G$ is a ring with local units.*

*Proof.* Let $T = \{w_1, \ldots, w_n\}$ be a finite set of elements of $(G * R) * G$. We may assume that $w_i = x_i v_{g_i} * h_i$, where $x_i \in R_{m_i}$. Then let $F = \{t \in G\colon t = m_i g_i \text{ or } t = g_i h_i \text{ for some } i\}$. Let $f = \sum_{t \in F} v_t$. Then $f * e$ is an idempotent in $(G * R) * G$ and for $w = xv_g * h \in T$,

$$\begin{aligned}(f * e)(xv_g * h)(f * e) &= \left(\left(\sum_{t \in F} v_t\right) xv_g * h\right)(f * e)\\ &= (xv_g * h)(f * e) \qquad \text{since } mg \in F\\ &= (xv_g)\left(\sum_{t \in F} v_{th^{-1}}\right) * h\\ &= xv_g * h \qquad \text{since } gh \in F.\end{aligned}$$

Therefore $(G * R) * G$ has local units.

We now wish to define a Morita context of the form $\{(G * R) * G, P = {}_{(G*R)*G}G * R_R, Q = {}_R G * R_{(G*R)*G}, R, [\ ,\ ], (\ ,\ )\}$. Since $G * R$ has a left and right $G$-action, $G * R$ is a left $(G * R) * G$-module under

$$(xv_g * h)(yv_t) = (xv_g)(^h(yv_t)) = (xv_g)(yv_{th^{-1}}) = xy_{ght^{-1}}v_{th^{-1}},$$

and a right $(G * R) * G$-module under

$$(xv_g)(yv_t * h) = ((xv_g)(yv_t))^h = xy_{gt^{-1}}v_{th}.$$

This action makes $G * R$ a unitary left $(G * R) * G$-module. For if $xv_h \in G * R$, then there exists $f = \sum_{g \in F} v_g$, $F$ a finite subset of $G$, such that $f(xv_h) = xv_h$. Then $(f * e)(xv_h) = xv_h$ and $G * R = ((G * R) * G)\, G * R$. Similarly $G * R$ is a unitary right $(G * R) * G$-module.

$G * R$ is a left and right $R$-module under $(xv_g)\, y = \sum_{h \in G} (xv_g)(yv_h) = \sum_{h \in G} xy_{gh^{-1}}v_h$ and $y(xv_g) = (yx)\, v_g$. Since $1(xv_g) = xv_g = (xv_g)\, 1$, both left and right module structures are unitary.

We verify that $G * R$ is a $(G * R) * G$-$R$-bimodule. The proof that it is an $R$-$(G * R) * G$-bimodule is similar.

$$\begin{aligned}
(xv_g * h)((yv_t)\, z) &= (xv_g * h)\left(\sum_{p \in G} yz_{tp^{-1}}v_p\right) \\
&= (xv_g)\left(\sum_{p \in G} yz_{tp^{-1}}v_{ph^{-1}}\right) \\
&= \sum_{p \in G} xy_{ght^{-1}}z_{tp^{-1}}v_{ph^{-1}} \\
&= \sum_{q \in G} xy_{ght^{-1}}z_{th^{-1}q}v_{q^{-1}} \\
&= (xy_{ght^{-1}}v_{th^{-1}})\, z \\
&= ((xv_g)(yv_{th^{-1}}))\, z \\
&= ((xv_g * h)(yv_t))\, z.
\end{aligned}$$

Therefore $G * R$ is a $(G * R) * G$-$R$-bimodule.

Now define $[\ ,\ ]\colon P \otimes_R Q \to (G * R) * G$ by

$$[xv_g, yv_h] = \sum_{p \in G} (xv_g * p)(yv_h * e) = \sum_{p \in G} xy_{gph^{-1}}v_{hp^{-1}} * p$$

for generators $xv_g$, $yv_h$ of $G * R$. We must show that $[\ ,\ ]$ is left and right $(G * R) * G$-linear and middle $R$-linear. We show that $[\ ,\ ]$ is right $(G * R) * G$-linear by checking that linearity holds for generators.

$$\begin{aligned}[xv_g, (yv_h)(zv_p * q)] &= [xv_g, yz_{hp^{-1}}v_{pq}] \\ &= \sum_{t\in G} x(yz_{hp^{-1}})_{gt(pq)^{-1}}v_{pqt^{-1}} * t \\ &= \sum_{t\in G} xy_{gtq^{-1}h^{-1}}z_{hp^{-1}}v_{pqt^{-1}} * t \\ &= \sum_{s\in G} xy_{gsh^{-1}}z_{hp^{-1}}v_{ps^{-1}} * sq \\ &= \left(\sum_{s\in G} xy_{gsh^{-1}}v_{hs^{-1}} * s\right)(zv_p * q) \\ &= [xv_g, yv_h](zv_p * q).\end{aligned}$$

The remaining linearity verifications for [ , ] are similar. Since for any generator $xv_g * h$ of $(G * R) * G$, $xv_g * h = [xv_g, v_{gh}]$, [ , ] is onto.

Define $( , ): Q \otimes_{(G*R)*G} P \to R$ by $(xv_g, yv_h) = xy_{gh^{-1}}$ for generators $xv_g, yv_h$ of $G * R$. We show that ( , ) is middle $(G * R) * G$-linear; the proofs of left and right $R$-linearity are similar.

$$\begin{aligned}(xv_g(yv_h * p), zv_q) &= (xy_{gh^{-1}}v_{hp}, zv_q) \\ &= xy_{gh^{-1}}z_{hpq^{-1}} \\ &= x(yz_{hpq^{-1}})_{gpq^{-1}} \\ &= (xv_g, yz_{hpq^{-1}}v_{qp^{-1}}) \\ &= (xv_g, (yv_h * p)\, zv_q).\end{aligned}$$

( , ) is also onto since for $x \in R$, $x = (xv_g, v_g)$.

We must still check that [ , ] and ( , ) satisfy the associativity conditions

$$[xv_g, yv_h]\, zv_p = xv_g(yv_h, zv_p)$$

and

$$(xv_g, yv_h)\, zv_p = xv_g[yv_h, zv_p].$$

We check that the first condition is satisfied; the second verification is similar.

$$\begin{aligned}[xv_g, yv_h]\, zv_p &= \left(\sum_{q\in G} xy_{gqh^{-1}}v_{hq^{-1}} * q\right)(zv_p) \\ &= \sum_{q\in G} xy_{gqh^{-1}}z_{hp^{-1}}v_{pq^{-1}}\end{aligned}$$

$$= \sum_{q \in G} x(yz_{hp^{-1}})_{gqp^{-1}} v_{pq^{-1}}$$

$$= \sum_{t \in G} (xv_g)(yz_{hp^{-1}} v_t)$$

$$= (xv_g)(yz_{hp^{-1}})$$

$$= (xv_g)(yv_h, zv_p).$$

Therefore $\{(G * R) * G, P, Q, R, [\ ,\ ], (\ ,\ )\}$ is a strict Morita context and we have

THEOREM 2.2. $(G * R) * G \cong M_G(R)^{\text{fin}}$, *the ring of matrices over R with rows and columns indexed by G and with finitely many nonzero entries.*

*Proof.* By [2, Theorems 2.1 and 2.2], $N = (G * R) * G \operatorname{End}_R(G * R) \cong (G * R) * G$, the largest left unitary $(G * R) * G$-submodule of $\operatorname{End}_R(G * R)$. For $\phi$ in $\operatorname{End}_R(G * R)$, $\phi$ is completely determined by its action on the $v_g$ since $\phi$ is left $R$-linear. Therefore, if we represent the elements of $\operatorname{End}_R(G * R)$ by matrices as in Theorem 1.3, $\operatorname{End}_R(G * R)$ is the ring of row finite matrices over $R$ with rows and columns indexed by $G$. If $\phi \in N$, then, since $N$ is unitary, $\phi = u\phi$ for some idempotent $u$ in $(G * R) * G$. Suppose $u = \sum_{i \in I} a_i * g_i$ with $a_i \in G * R$, $g_i \in G$. Let $F = \{t \in G: v_t a_i \neq 0 \text{ for some } a_i\}$, and $f = \sum_{t \in F} v_t$. Then $(f * e)\, u = u$. Therefore $\phi = (f * e)\, \phi$ and $(v_g)\, \phi = (\sum_{t \in F} v_g v_t)\, \phi = 0$ if $g \notin F$. Thus only finitely many rows of $\phi$ are nonzero so that $\phi \in M_G(R)^{\text{fin}}$ and $N \subset M_G(R)^{\text{fin}}$.

To show equality, we must show that $M_G(R)^{\text{fin}}$ is a unitary $(G * R) * G$-module. Clearly $M_G(R)^{\text{fin}}$ is a $(G * R) * G$-submodule of $\operatorname{End}_R(G * R)$. Suppose $\chi \in M_G(R)^{\text{fin}}$, and let $L$ be the finite set $\{g \in G: v_g \chi \neq 0\}$. Then $\chi = (\sum_{g \in L} v_g * e)\, \chi$ and $M_G(R)^{\text{fin}}$ is unitary.

It is interesting to note that the Morita context $\{(G * R) * G, P, Q, R, [\ ,\ ], (\ ,\ )\}$ defined above coincides with that of Cohen [6] or [7] if $G$ is finite. For then $G * R$ has a unit and $(G * R)^G = \{\sum_{g \in G} xv_g : x \in R\} \cong R$. One might ask if a more general Morita context exists for rings with $G$-action. The Morita context in Section 1 is strict if and only if $R$ is strongly $G$-graded. What is the analogue to strong $G$-grading for rings with $G$-action?

## ACKNOWLEDGMENT

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada.

## References


1. G. D. Abrams, Morita equivalence for rings with local units, *Comm. Algebra* **11** (1983), 801–837.
2. P. N. Anh and L. Marki, Morita equivalence for rings without identity, *Tsukuba J. of Math.*, to appear.
3. H. Bass, "Lectures on Topics in Algebraic $K$-Theory," Tata Institute for Fundamental Research, Bombay, 1967.
4. M. Beattie, A generalization of the smash product of a graded ring, *J. Pure Appl. Alg.*, to appear.
5. R. J. Blattner and S. Montgomery, A duality theorem for Hopf module algebras, *J. Algebra* **95** (1985), 153–172.
6. M. Cohen, A Morita context related to finite automorphism groups of rings, *Pacific J. Math.* **98** (1982), 37–54.
7. M. Cohen and D. Fishman, Hopf algebra actions, *J. Algebra* **100** (1986), 363–379.
8. M. Cohen and S. Montgomery, Group-graded rings, smash products, and group actions, *Trans. Amer. Math. Soc.* **282** (1984), 237–258.
9. C. Nastasescu and F. Van Oystaeyen, "Graded Ring Theory," North-Holland Mathematical Library Vol. 28, North-Holland, Amsterdam, 1982.
10. D. Quinn, Group-graded rings and duality, *Trans. Amer. Math. Soc.* **292** (1985), 155–167.